ON A VARIATIONAL INEQUALITY FORMULATION OF A ONE DIMENSIONAL SINGLE-PHASE MELTING PROBLEM.

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Abstract

A single – phase melting moving boundary problem is formulated after a change of variable, as an parabolic variational inequality.

Introduction

The term moving boundary problems (MBP'S) is commonly used when the boundary is associated with time dependent problems and the boundary of the domain is not known in advanced but has to be determined as a function of time and space.

Heat conduction or diffusion type problems with phase change constitute a large class of (MBP'S) usually called Stefan problems. Moving boundary problems have received much attention in recent year due to their practical importance in engineering and science [1], [2], [3].

The purpose of this paper is to obtain a variational inequality formulation of a single-phase melting problem.

Description of the problem

The problem concerns heat transfer in an ice-water medium occupying the region $0 \le X \le 1$. At any time t, the water that under going phase change, is contained in the region $\left(0 < x < s(t)\right)$ and the rest of the region outside it, is occupied by ice, as in [4]. The temperature of the water is assumed to be equal to zero which is also the critical temperature of phase change.

Mathematical Formulation of the problem

The problem is expressed in non-dimensional form. If u(x,t) denotes the temperature distribution at distance x and time t, s(t) is the position of the interface at time t, then the problem is to find the pair of unknowns u(x,t) and s(t). The problem may be defined mathematically by the following equation and conditions [5].

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \qquad 0 < x < s(t), t > 0 \cdots\cdots\cdots (1)$$

the boundary conditions are given by :-

u=1 , x=0 , t > 0 ....(2)
u=0 , x>0 , t=0 ....(3)

u=0

$$\frac{-\partial u}{\partial x} = \lambda \frac{ds}{dt} \qquad x = s(t)\, , t > 0 \text{ ....(4)}$$

S(0) = 1 ....(5)

Where λ is "Latent heat" and (4) is known as the "Stefan condition".

Variational Inequality Formulation of the problem

Another possibility is to formulate moving boundary problems in terms of certain inequalities, either of a differential or variational nature.

In either case the expressions refer to a fixed domain and explicit use of the Stefan or other interface condition is avoided. A variational formulation is only possible after some suitable transformation has removed the discontinuity. Such a transformation was first proposed by Duvaut (1973) for moving boundaries prompted by an idea of Baiocchi (1971) for stationary, free boundaries. Fremoud (1974, 1980) called the new variable , the freezing index.

For the single-phase melting problem defined by equations (1-5), the Duvaut transformation, or the Fremond freezing index, is

$$w(x,t)=\int_{t(x)}^{t} u(x,\tau)d\tau \qquad ,0\le x\le s(t)$$

$$w(x,t)=0 \qquad , s(t)\le x\le 1 \qquad \text{.................}(6)$$

where was have introduced the fixed, finite domain $0\le x\le 1$, and where $t=\ell(x)$ define the time when the ice at point x changes to water, i.e. $\ell(x)$ is the time when the melting boundary is at x. It is the inverse of s(t) so that $\ell^{-1}(t)=s(t)$.

To show that w(x,t) and its first derivative $\frac{\partial w}{\partial x}$ are continuous on the melting boundary and throughout the range $0\le x\le 1$, we have form (6) that:-

$$\frac{\partial w}{\partial x}=\int_{\ell(x)}^{t}\frac{\partial u}{\partial x}(x,\tau)d\tau-\frac{\partial \ell}{\partial x}u(x,\ell(x)) \qquad 0\le x\le s(t)$$

$$\frac{\partial w}{\partial x}=0 \qquad s(t)\le x\le 1 \qquad \cdots\cdots\cdots\cdots\cdots\cdots(7)$$

since the temperature u is zero on the interface x=s(t) , i.e, u(x,ℓ(x))=0, then (7), becomes :-

$$\frac{\partial w}{\partial x}=\int_{\ell(x)}^{t}\frac{\partial u}{\partial x}(x,\tau)d\tau \qquad ,0\le x\le s(t)$$

$$\cdots\cdots\cdots\cdots(8)$$

$$\frac{\partial w}{\partial x}=0 \qquad ,s(t)\le x\le 1$$

and form (8) we know that $\frac{\partial w}{\partial x}=0$ on x=s(t).

Furthermore ,

$$\frac{\partial^2 w}{\partial x^2}=\int_{\ell(x)}^{t}\frac{\partial^2 u}{\partial x^2}d\tau-\frac{\partial \ell}{\partial x}\frac{\partial u}{\partial x}(x,\ell(x)), \qquad 0\le x\le s(t)\cdots\cdots\cdots\cdots\cdots(9)$$

Noting that $\frac{\partial \ell}{\partial x} = \frac{-1}{\frac{\partial s}{\partial t}}$

So that the Stefan condition (4) implies

$\frac{\partial u}{\partial x}\frac{\partial \ell}{\partial x} = \lambda$, then the equation (9) with use of (1) becomes :-

$$\frac{\partial^2 w}{\partial x^2} = \int_{t(x)}^{t} \frac{\partial u}{\partial t} d\tau - \lambda = \frac{\partial}{\partial t}\int_{t(x)}^{t} u d\tau - \lambda = \frac{\partial w}{\partial t} - \lambda$$

Thus the transformed equation is :-

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \lambda \qquad ,0 \le x \le s(t)$$

$$\cdots\cdots\cdots\cdots\cdots(10)$$

$$w=0 \qquad , s(t) \le x \le 1$$

with w(x,t) > 0 in $0 \le x \le s(t)$.

Thus w(x,t) and its first derivative are continuous throughout the domain $0 \le x \le 1$, then we have the inequalities :-

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \lambda \ge 0 \qquad ,w \ge 0, 0 \le x \le 1 \cdots\cdots\cdots\cdots(11)$$

and $\left(\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \lambda\right)w = 0 \qquad ,0 \le x \le 1 \cdots\cdots\cdots\cdots(12)$

Avariational form of (11) and (12) is derived by multiplying the left-hand side of the first of (110 by (v-w) using the test functions v which are non-negative in 0<x<1 with v=0 on x=1, and for which v and $\frac{\partial v}{\partial x}$ are square integrable in 0<x<1, i.e

$\int_0^1 v^2 dx \quad and \int_0^1 \left(\frac{\partial v}{\partial x}\right)^2 dx$ are bounded

Thus

$$\int_0^1 \left(\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \lambda\right)(v-w)dx$$

$$= \int_0^1 \frac{\partial w}{\partial t}(v-w)dx - \left[(v-w)\frac{\partial w}{\partial x}\right]_0^1 + \int_0^1 \frac{\partial w}{\partial x}\frac{\partial}{\partial x}(v-w)dx - \int_0^1 \lambda(v-w)dx$$

Remembering that v=w=0, x=1, and $\frac{\partial w}{\partial x} = 0$, x=0, together with use of (12), we obtain after slight rearrangement :-

$$\lambda\int_0^1 (v-w)dx + \int_0^1 \left(\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \lambda\right)v dx.$$

But the final term on the right-hand side of this equation is non-negative since $v \geq 0$, and the first of (11) is to be satisfied.
We therefore have the variational inequality

$$\int_0^1 \frac{\partial w}{\partial t}(v-w)dx + \int_0^1 \frac{\partial w}{\partial x}\frac{\partial}{\partial x}(v-w)dx \geq \lambda \int_0^1 (v-w)dx \cdots\cdots\cdots\cdots (13)$$

Define the inner products (.,.) and a(.,.) with reference to a domain D in two space dimensions as follows :-

$$a(u, v-u) = \iint_D \nabla u \nabla (v-u) dx\, dy \cdots\cdots\cdots\cdots\cdots (14)$$

$$= \iint_D \{u_x(v_x - u_x) + u_y(v_y - u_y)\} dx\, dy$$

and $(f, v-u) = \iint_D f(v-u)dx\, dy \cdots\cdots\cdots\cdots\cdots\cdots (15)$

using the nomenclature of (14) and (15) , the inequality formulation of the melting problem (13), equivalent to the differential forms (11) and (12), becomes

$$(w_t, v-w) + a(w, v-w) \geq (\lambda, v-w) \text{..................}(16)$$

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الخلاصة

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